Universal charge-radius relation for subatomic and astrophysical compact objects

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Electron-positron pair creation in supercritical electric fields limits the net charge of any static, spherical object, such as superheavy nuclei, strangelets, and Q-balls, or compact stars like neutron stars, quark stars, and black holes. For radii between 4×10^2 fm and 10^4 fm the upper bound on the net charge is given by the universal relation $Z = 0.71 R_{\rm fm}$, and for larger radii (measured in fm or km) $Z = 7 \times 10^{-5} R_{\rm fm}^2 = 7 \times 10^{31} R_{\rm km}^2$. For objects with nuclear density the relation corresponds to $Z \approx 0.7 A^{1/3}$ ($10^8 < A < 10^{12}$) and $Z \approx 7 \times 10^{-5} A^{2/3}$ ($A > 10^{12}$), where A is the baryon number. For some systems this universal upper bound improves existing charge limits in the literature.

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Spontaneous formation of real electron-positron pairs in supercritical electric fields is known to lead to screening of highly charged objects [1, 2]. This is in addition to the vacuum polarization effect caused by virtual pairs [2]. In the following it will be shown, that pair formation implies a universal upper limit on the net charge of any static, spherical object of given radius. The upper bound on the net charge for $R > 4 \times 10^2$ fm is $Z_\infty = 0.71 R_{\rm fm} = 7.1 \times 10^{17} R_{\rm km}$, where the radius is measured in fm or km to show the typical scales for subatomic objects or compact stars like neutron stars, quark stars and black holes, and the subscript ∞ indicates infinite time available for pair formation. Hypothetical superheavy nuclei, quark nuggets (also known as strangelets), and other objects with nuclear matter density have $R_{\rm fm} \approx A^{1/3}$, where A is the baryon number, so the charge bound can be recast as $Z_\infty \approx 0.7 A^{1/3}$. This universal charge bound is complementary to and often significantly more restrictive than other bounds in the literature. However, it implicitly assumes that infinite time is available to populate electron levels via pair creation. Taking the relevant timescales into account the maximum charge increases for $R > 10^4$ fm ($A > 10^{12}$) to $Z \approx 7 \times 10^{-5} R_{\rm fm}^2 \approx 7 \times 10^{-5} A^{2/3}$ (the latter expression assuming nuclear matter density). This relation improves existing limits for e.g. strangelets, reproduces an earlier result for massive black holes, but is inferior to charge limits based on stellar stability for gravitationally bound neutron stars when only finite time is available for pair creation.

In the following the upper bound on the charge of spherical objects will be derived in the context of a relativistic Thomas-Fermi model calculation following the approach of Müller and Rafelski [1]. The universal relation will be analytically derived in the high mass, infinite time limit, and as typical examples homogeneously charged superheavy nuclei and color-flavor locked strangelets dominated by surface charge will be shown numerically to approach the universal charge-radius relation. Finite time for pair creation will then be taken into account and the corresponding (higher) charge limits derived. The results will be compared to earlier charge-mass relations for strangelets, and to limits for astrophysical objects. Whereas the derivations do not include general relativity it is argued, that the universal charge-radius limit should also apply approximatively to black holes, and in fact the results agree with a limit derived for black holes in general relativity.

In a continuous approximation at zero temperature (more about finite temperature effects later) the number density of electrons, $n_{\rm e}$, is given by the electron mass, $m_{\rm e}$, and Fermi energy, $\mu_{\rm e}$, via

$$n_{\rm e} = \frac{\left[\left(\mu_{\rm e}^{\rm eff} + e\phi\right)^2 - m_{\rm e}^2\right]^{3/2}}{3\pi^2} \theta\left(\mu_{\rm e}^{\rm eff} + e\phi - m_{\rm e}\right). \quad (1)$$

Here the effective chemical potential $\mu_{\rm e}^{\rm eff} = \mu_{\rm e} - e\phi$, where ϕ is the electric potential, and the θ-function takes into account that $\mu_{\rm e}$ must exceed the electron mass. In the usual Thomas-Fermi model describing neutral atomic systems one takes the energy needed to add an additional electron, $\mu_{\rm e}^{\rm eff} = m_{\rm e}$. In the present context the focus is on maximally charged rather than neutral systems, and here one instead takes $\mu_{\rm e}^{\rm eff} = -m_{\rm e}$, corresponding to the top of the negative energy sea [1]. This represents a situation where all levels accessible to spontaneous vacuum decay are filled, and as shown in [1, 2] it reproduces the results of single-particle calculations carried out for core charges up to a few hundred that demonstrate how more and more real electron states dive into the negative energy continuum as the core charge increases, leading to the creation of a negatively charged vacuum. Thus

$$n_{\rm e} = \frac{\left[(e\phi)^2 - 2m_{\rm e} e\phi\right]^{3/2}}{3\pi^2} \theta\left(e\phi - 2m_{\rm e}\right). \quad (2)$$

In regions of space containing only electrons Poisson's equation is given by $\nabla^2(e\phi) = 4\pi e^2 n_{\rm e}$, or in dimensionless units, where the radial coordinate $\xi \equiv m_{\rm e} r$, and the

normalized potential $y \equiv e\phi/m_{\rm e}$, with $\alpha \equiv e^2$,

$$\nabla^2 y = \frac{4\alpha}{3\pi}\left[y^2 - 2y\right]^{3/2}\theta(y-2). \tag{3}$$

In regions with additional charge, a corresponding source term must be added to the right-hand side of Poisson's equation. Two illustrative types of core charge distributions have been considered: A uniformly charged spherical core (like a nucleus) and a uniformly charged spherical shell with no internal charge (like a color-flavor locked quark matter lump or a large, perfectly conducting nuclear matter system, where the net core charge moves to the surface). Together these idealized distributions span the likely range of real distributions, and as shown in the following, they lead to identical results for the screened charge in the limit of large system radius.

Boundary conditions for Poisson's equation are $\nabla y \to 0$ for $\xi \to 0$ and $y \to 0$ for $\xi \to \infty$. But the actual behavior of y for large ξ is explicitly known in the maximally charged case. Denote the total (positive) core charge by $Z_{\rm core}$ and the total number of electrons by $N_{\rm e}$. Then $Z_\infty = Z_{\rm core} - N_{\rm e} > 0$ is the net charge of the maximally charged system seen by an observer outside the radius ξ_2, where the electron density drops to zero, $y(\xi_2) = 2$ (corresponding to $e\phi = 2m_{\rm e}$). But to this observer

$$y = \frac{Z_\infty \alpha}{\xi} \text{ for } \xi \geq \xi_2. \tag{4}$$

Using this relation for $\xi = \xi_2$ gives

$$Z_\infty = \frac{2m_{\rm e}r_2}{\alpha}, \tag{5}$$

where r_2 denotes the physical radius corresponding to $\xi = \xi_2$. In the bulk of a homogeneously charged core electrons neutralize the local core charge, and a deviation from local charge neutrality occurs only very close to the surface. As confirmed by numerical solutions of Poisson's equation the characteristic width of the screening electron cloud outside the core charge is of order $m_{\rm e}^{-1}$ or a few hundred fm (deviations from local charge neutrality occurs within a similar zone inside the surface). Thus in the limit of $r_2 \gg m_{\rm e}^{-1}$ the width is small compared to the physical radius of the core, R, and to good approximation the net charge of a maximally charged spherical system with $R \gg m_{\rm e}^{-1} \approx 4\times 10^2$ fm is therefore (since $R \approx r_2$)

$$Z_\infty \approx \frac{2m_{\rm e}R}{\alpha} = 0.71 R_{\rm fm} = 7.1\times 10^{17} R_{\rm km}. \tag{6}$$

For objects with density of order nuclear matter density $R_{\rm fm} \approx A^{1/3}$. Therefore, for such objects charge and baryon number are related by

$$Z_\infty \approx 0.7 A^{1/3} \text{ for } A \gg 10^8. \tag{7}$$

Figure 1 shows how well the universal charge-radius relation fits numerical solutions to the relativistic Thomas-Fermi model at large radius. The example here is for color-flavor locked strangelets composed of a core of equal numbers of up, down, and strange quarks, leading to zero quark charge density in the bulk of the core, but with a core surface charge due to surface depletion of strange quarks, so that $Z_{\rm core} \approx 0.3A^{2/3}$ [3] and radius $R_{\rm fm} \approx 1.1A^{1/3}$ (such a system has a discontinuity in $dy/d\xi$ at the core surface in addition to the boundary conditions previously described for the Poisson equation). One clearly sees how the actual solution of Poisson's equation for the net screened charge shifts from $Z_\infty \approx Z_{\rm core}$ to the universal relation $Z_\infty \approx 2\xi/\alpha$ when going from small ($\xi \ll 1$) to large ($\xi \gg 1$) radii. Similar behavior is found for systems with other $Z_{\rm core}(A)$ dependence and regardless of whether the core charge is uniformly distributed, or distributed as a surface charge as in the case of strangelets and for cores that behave as ideal conductors. In all cases tested, the maximal charge approaches the *same* universal charge-radius relation $Z_\infty \approx 0.71 R_{\rm fm}$ for $m_{\rm e}R \equiv \xi \gg 1$ (physical radius R exceeding 4×10^2 fm) as expected.

So far the calculations have assumed a static situation with infinite time available to fill all accessible electron states. However the pair formation process involves tunneling and the rate for this (number of pairs produced per volume per time) has been shown to be [4]

$$W = \frac{m_{\rm e}^4}{4\pi^3}\left(\frac{E}{E_{\rm cr}}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-n\pi\frac{E_{\rm cr}}{E}\right), \tag{8}$$

where the critical field is $E_{\rm cr} = m_{\rm e}^2/\alpha^{1/2} \approx 1.3\times 10^{16}$V/cm. Taking $E = Z\alpha^{1/2}/R^2$ to represent the "surface region" of the spherical charge distribution studied here, and including only the leading $n=1$ term in the sum, this corresponds to

$$W \approx \frac{m_{\rm e}^4\alpha^2 Z^2}{4\pi^3\xi^4}\exp\left(-\pi\frac{\xi^2}{\alpha Z}\right). \tag{9}$$

Taking the active charge producing layer to have a volume $V = 4\pi R^2\Delta R$, with thickness $\Delta R \approx m_{\rm e}^{-1}$, assuming $WV = -dZ/dt$, and defining a characteristic timescale τ for charge equilibration from $dZ/Z \equiv -dt/\tau$, this leads to a timescale

$$\tau = \frac{\pi}{\alpha m_{\rm e}} x\exp(x) = 5.5\times 10^{-19}\text{s}\, x\exp(x), \tag{10}$$

where $x \equiv \pi\xi^2/\alpha Z$. Because of the exponential a finite time available for pair production corresponds to a (roughly) fixed value of x for a wide range of τ (it also means that the results do not depend crucially on the assumptions about V; taking $V = 4\pi R^3/3$ instead of the surface layer chosen makes little difference to the charge relations below). In other words, pair production in a finite time gives a maximum charge value which is proportional to ξ^2, rather than ξ as were the case with infinite

time available. For example, for $\tau = 1$ s the maximally allowed charge becomes

$$Z_{1s} \approx 11.2\xi^2 \approx 7 \times 10^{-5} R_{\rm fm}^2 \approx 7 \times 10^{-5} A^{2/3}, \quad (11)$$

where the last equality assumes nuclear matter density. For $\tau = 10^{10}$ years one gets very similar results, except for dividing the numerical prefactors by 2.0, whereas a typical weak interaction timescale of 10^{-10} s corresponds to multiplication of the numerical prefactors by 2.4. Concentrating on Z_{1s} we therefore have two regimes for the universal charge-radius relation. For $Z_{1s}(\xi) < Z_\infty(\xi)$ pair production is sufficiently rapid to fill the electron states as assumed in the relativistic Thomas-Fermi model, and therefore $Z \approx Z_\infty \approx 2\xi/\alpha$ for $1 < \xi < 25$. For $Z_{1s}(\xi) > Z_\infty(\xi)$ ($\xi > 25$) we have instead $Z \approx Z_{1s} \approx 11.2\xi^2$. The corresponding division between the two regimes in terms of radius and baryon number (assuming nuclear matter density) is $R_{\rm fm} \simeq 1.0 \times 10^4$ and $A \approx 10^{12}$ respectively. The higher charge permitted for macroscopic spheres because of the finite time effect explains why a Van de Graaff generator can work at potential in excess of 1 MV as would be the limit with infinite time to screen the charge with electrons from pair production.

The existence of a universal maximal charge-radius relation for spherical objects regardless of their physical nature is interesting in its own right, but it also has applications to several areas of subatomic physics and astrophysics as outlined in the following.

It has been speculated that there could exist branches of metastable superheavy nuclei. If so, the charge of the maximally ionized superheavy ion should obey the relation $Z \approx 0.7A^{1/3}$ ($10^8 < A < 10^{12}$) or $Z \approx 7 \times 10^{-5} A^{2/3}$ ($A > 10^{12}$). Because the nuclear charge increases almost linearly with A, charge screening becomes important already for $A \approx 10^3$ (as also pointed out in [1]), but numerical solutions of Poisson's equation show that one needs $\xi > 1$ or $A > 10^8$ for the universal relation to become quantitatively accurate.

The possible existence of metastable or even stable quark nuggets or strangelets has been widely discussed [5]. For large chunks of quark matter, which could exist in our Galaxy as a result of binary compact star collisions, the (Z, A)-relation in the high-mass limit has so far been taken as either $Z \approx 8A^{1/3}$ (for non-color-flavor locked strangelets with $A \lesssim 10^7$ [6]), or $Z \approx 0.3A^{2/3}$ (for color-flavor locked strangelets regardless of A [3]). The possible importance of charge screening due to supercritical field electrons was mentioned by Farhi and Jaffe [7], and a first numerical study was included in [8]. From the discussion above it follows that the universal relation $Z \approx 0.7A^{1/3}$ ($10^8 < A < 10^{12}$) and $Z \approx 7 \times 10^{-5} A^{2/3}$ ($A > 10^{12}$) can be applied for $A > 10^8$ as an upper envelope on the strangelet charge regardless of the details of the quark phase, and for color-flavor locked quark matter this envelope represents the actual maximum ionization state possible, since the "envelope" value of Z is smaller than the core charge $0.3A^{2/3}$.

Q-balls have been suggested in various varieties, some of which can be charged [9]. Again, the universal relation derived here should apply.

Turning to astrophysical objects, it is normally assumed that compact stars such as neutron stars and quark stars are close to electrical neutrality, since a net positive charge would attract electrons from the interstellar medium. However, many papers have dealt with the theoretical possibility of non-zero charge and its influence on stellar structure, and limits (typically from arguments related to stellar stability) have been placed on the charge allowed. From a comparison of gravitational binding and electric repulsion, several authors (e.g. [10]) have found that the net charge of gravitationally bound stars is limited by $Z < 10^{-36}A$, where the baryon number $A \approx 10^{57}$ for a typical compact star, thus $Z < 10^{21}$. From the universal relation derived above, we have the limit $Z_{1s} < 7 \times 10^{-5} A^{2/3}$, which is conceptually very different, but happens to give a larger number for $A \approx 10^{57}$, namely $Z_{1s} < 7 \times 10^{33}$. This shows that stabilization of charged neutron stars should result from trapping of real electrons from the surroundings; there is not enough time to ensure sufficient numbers of electrons from pair creation. With infinite time available, the static maximal charge would be $Z_\infty \approx 0.7A^{1/3} \approx 10^{19}$, i.e. stable with respect to the electric repulsion, but to approach this limit requires orders of magnitude more time than the age of the Universe if one were to start out with a much higher charge.

Strange stars consisting of quark matter are self-bound due to strong interactions in addition to gravity, so the relevant stability limit here comes from a comparison of the Coulomb energy and the total (strong interaction) binding energy per baryon. Stability requires $Z^2\alpha/AR$ to be less than a few MeV, which is easily satisfied (by 8 orders of magnitude) for $Z = Z_{1s}$. Therefore the charge of strange stars may in principle saturate the universal limit $Z_{1s} \approx 7 \times 10^{-5} A^{2/3}$, and such a system could have interesting properties related to pair creation even at zero temperature in addition to the (much faster) finite temperature pair creation phenomena discussed in [11].

Black holes are formally exempt from the treatment above since they are described by general relativity. However, formation of an astrophysical black hole involves the collapse of a mass (and for the present purpose charge) that seen from an outside observer only asymptotically reaches the horizon. Therefore one might expect that the universal charge-radius relation would indeed be obeyed by the black hole because it would be obeyed during the formation process, and the maximal net charge seen from the outside again should not exceed $7 \times 10^{31} R_{\rm km}^2$. Studies of pair formation in a general relativistic description of black holes has in fact led to results almost identical to this [12], a result which is many orders of magnitude

below other limits derived for the maximal charge of a stable black hole as long as the mass is below 10^8 solar masses [12].

The derivations above were all based on zero temperature relations. These relations remain valid for temperature $T \ll \mu_\mathrm{e} - m_\mathrm{e} = e\phi - 2m_\mathrm{e}$, so at low temperatures thermal electrons can be neglected except near the edge of the zero temperature maximally charged configuration, where $e\phi \to 2m_\mathrm{e}$. The lowest order thermal contribution to the electron number density in this regime is roughly $m_\mathrm{e}^{3/2}T^{3/2}$, giving a thermal contribution to the total charge which is suppressed by $(T/m_\mathrm{e})^{3/2}$ relative to the zero temperature charge $Z_{1\mathrm{s}}$. Therefore the zero temperature relations provide good approximations for $T \ll m_\mathrm{e}$.

Another caveat in applying the universal charge-radius relation derived above to real subatomic or astrophysical objects is related to the tunneling timescale involved. The larger the system, the lower is the probability of filling all electron levels on a short timescale. Therefore, even though the universal relation is independent of the internal composition and structure of the spherical core, the realization of the universal behavior in realistic systems will depend somewhat on the nature of the object and its previous history. Systems with a high electrical conductivity (e.g. nuclear matter or quark matter) will rearrange any initial net charge in a way so that the charge gets concentrated in a thin surface layer. The small width of this layer and the surrounding electron atmosphere improves chances for the universal charge-radius relation to be realized. Other systems, such as strangelets or quark stars composed of color-flavor locked quark matter, are electrically neutral in the bulk quark phase already [13], and have only a net quark charge on the surface [3, 14], so here the electron shield is thin almost "by construction". Positively charged objects in general tend to neutralize by trapping electrons from the surroundings, e.g. from the interstellar medium. Therefore, most of the bulk of macroscopic objects will be close to charge neutral, and the interesting physics will be related to processes ionizing such objects. Such ionization will take place "from the outside" and involve a restricted radial range, again minimizing the timescale problem.

In conclusion, it has been demonstrated that electrons created in supercritical fields lead to a universal relation between the maximal charge and radius of any static, spherically symmetric object with a size exceeding a few hundred fm. The relation limits the charges of objects such as superheavy nuclei, strangelets, Q-balls, neutron stars, quark stars, and black holes. For some of the objects studied, the new limits are more restrictive than other charge relations and limits existing in the literature.

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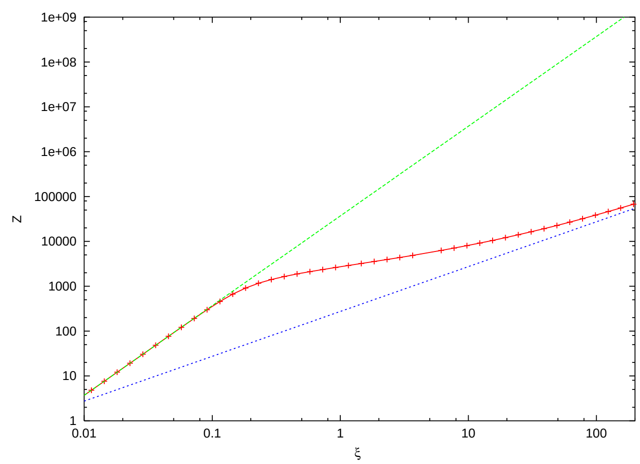

FIG. 1: Charge as a function of scaled radius, ξ. The lower (blue, dotted) line represents the universal charge-radius relation $Z_\infty \approx 2\xi/\alpha$ valid for large radii and infinite time. The upper (green, dashed) line is the quark charge of color-flavor locked strangelets, and the middle points with fitted (red) curve shows the maximal net charge, Z, as calculated numerically from the relativistic Thomas-Fermi model. As can be clearly seen, the electron screening due to the supercritical electric field becomes important for $\xi > 0.1$, and the universal charge-radius relation is a good representation for $\xi \gg 1$.